

Rosenthal's inequalities for independent and negatively dependent random variables under sub-linear expectations with applications

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Abstract Classical Kolmogorov's and Rosenthal's inequalities for the maximum partial sums of random variables are basic tools for studying the strong laws of large numbers. In this paper, motivated by the notion of independent and identically distributed random variables under the sub-linear expectation initiated by Peng [12] [14], we introduce the concept of negative dependence of random variables and establish Kolmogorov's and Rosenthal's inequalities for the maximum partial sums of negatively dependent random variables under the sub-linear expectations. As an application, we show that Kolmogorov's strong law of larger numbers holds for independent and identically distributed random variables under a continuous sub-linear expectation if and only if the corresponding Choquet integral is finite.

Keywords sub-linear expectation, capacity, Kolmogorov's inequality, Rosenthal's inequality, negative dependence, strong laws of large numbers

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1 Introduction and notations.

Non-additive probabilities and non-additive expectations are useful tools for studying uncertainties in statistics, measures of risk, superhedging in finance and non-linear stochastic calculus, c.f. Denis and Martini [2], Gilboa [5], Marinacci [7], Peng [10] [11] [12] [13] etc. This paper considers the general sub-linear expectations and related non-additive probabilities generated by them. The notion of independent and identically distributed random variables under the sub-linear expectations is introduced by Peng [12] [14] and the weak convergence such as central limit theorems and weak laws of large numbers are studied. Because the proofs of classical Kolmogorov's inequalities and Rosenthal's inequalities for the maximum partial sums of random variables depend basically on the additivity of the probabilities and the expectations, such inequalities have not been established under the sub-linear expectations. As a result, very few results on strong laws of larger numbers are found under the sub-linear expectations. Recently, Chen [1] obtained Kolmogorov's strong law of larger numbers for i.i.d. random variables under the condition of finite $(1 + \epsilon)$ -moments by establishing an inequality of an exponential moment of partial sums of truncated independent random variables. The moment condition is much stronger than the one for the classical Kolmogorov strong law of larger numbers. Also, Gao and Xu [3] [4] studied the large deviations and moderate deviations for quasi-continuous random variables in a complete separable metric space under the Choquet capacity generalized by a regular sub-linear expectation. The main purpose of this paper is to establish basic inequalities for the maximum partial sums of independent random variables in the general sub-linear expectation spaces. These inequalities are basic tools to study the strong limit

theorems. They are also essential tools to prove the functional central limit theorem (c.f. Zhang [24]). In the remainder of this section, we give some notations under the sub-linear expectations. For explaining our main idea, we prove Kolmogorov's inequality as our first result. And then, we introduce the concept of negative dependence under the sub-linear expectation which is an extension of independence as well as the classical negative dependence. In the next section, we establish Rosenthal's inequalities for this kind of negatively dependent random variables. In Section 3, as applications of these inequalities, we establish the Kolmogorov type strong laws of large numbers under the weakest moment conditions. In particular, we show that Kolmogorov's type strong law of large numbers holds for independent and identically distributed random variables under a continuous sub-linear expectation if and only if the corresponding Choquet integral is finite.

We use the notations of Peng [14]. Let (Ω, \mathcal{F}) be a given measurable space and let \mathcal{H} be a linear space of real functions defined on (Ω, \mathcal{F}) such that if $X_1, \dots, X_n \in \mathcal{H}$ then $\varphi(X_1, \dots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_n,$$

for some $C > 0, m \in \mathbb{N}$ depending on φ .

\mathcal{H} is considered as a space of "random variables". In this case we denote $X \in \mathcal{H}$.

Remark 1.1. It is easily seen that if $\varphi_1, \varphi_2 \in C_{l,Lip}(\mathbb{R}_n)$, then $\varphi_1 \vee \varphi_2, \varphi_1 \wedge \varphi_2 \in C_{l,Lip}(\mathbb{R}_n)$ because $\varphi_1 \vee \varphi_2 = \frac{1}{2}(\varphi_1 + \varphi_2 + |\varphi_1 - \varphi_2|)$, $\varphi_1 \wedge \varphi_2 = \frac{1}{2}(\varphi_1 + \varphi_2 - |\varphi_1 - \varphi_2|)$.

Definition 1.2. A **sub-linear expectation** $\widehat{\mathbb{E}}$ on \mathcal{H} is a functional $\widehat{\mathbb{E}} : \mathcal{H} \rightarrow \overline{\mathbb{R}} := [-\infty, \infty]$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

- (a) **Monotonicity:** If $X \geq Y$ then $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$;
- (b) **Constant preserving :** $\widehat{\mathbb{E}}[c] = c$;
- (c) **Sub-additivity:** $\widehat{\mathbb{E}}[X + Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ whenever $\widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;
- (d) **Positive homogeneity:** $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X]$, $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a sub-linear expectation space. Give a sub-linear expectation $\widehat{\mathbb{E}}$, let us denote the conjugate expectation $\widehat{\mathcal{E}}$ of $\widehat{\mathbb{E}}$ by

$$\widehat{\mathcal{E}}[X] := -\widehat{\mathbb{E}}[-X], \quad \forall X \in \mathcal{H}.$$

Obviously, for all $X \in \mathcal{H}$, $\widehat{\mathcal{E}}[X] \leq \widehat{\mathbb{E}}[X]$. We also call $\widehat{\mathbb{E}}[X]$ and $\widehat{\mathcal{E}}[X]$ the upper-expectation and lower-expectation of X respectively.

Definition 1.3. (Peng [12] [14])

- (i) **(Identical distribution)** Let \mathbf{X}_1 and \mathbf{X}_2 be two n -dimensional random vectors defined respectively in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $\mathbf{X}_1 \stackrel{d}{=} \mathbf{X}_2$ if

$$\widehat{\mathbb{E}}_1[\varphi(\mathbf{X}_1)] = \widehat{\mathbb{E}}_2[\varphi(\mathbf{X}_2)], \quad \forall \varphi \in C_{l,Lip}(\mathbb{R}_n),$$

whenever the sub-expectations are finite.

- (ii) **(Independence)** In a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, a random vector $\mathbf{Y} = (Y_1, \dots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent to another random vector $\mathbf{X} = (X_1, \dots, X_m)$, $X_i \in \mathcal{H}$ under $\widehat{\mathbb{E}}$ if for each test function $\varphi \in C_{l,Lip}(\mathbb{R}_m \times \mathbb{R}_n)$ we have

$$\widehat{\mathbb{E}}[\varphi(\mathbf{X}, \mathbf{Y})] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(\mathbf{x}, \mathbf{Y})]_{\mathbf{x}=\mathbf{X}}],$$

whenever $\overline{\varphi}(\mathbf{x}) := \widehat{\mathbb{E}}[|\varphi(\mathbf{x}, \mathbf{Y})|] < \infty$ for all \mathbf{x} and $\widehat{\mathbb{E}}[|\overline{\varphi}(\mathbf{X})|] < \infty$.

- (iii) (**IID random variables**) A sequence of random variables $\{X_n; n \geq 1\}$ is said to be independent, if X_{i+1} is independent to (X_1, \dots, X_i) for each $i \geq 1$. It is said to be identically distributed, if $X_i \stackrel{d}{=} X_1$ for each $i \geq 1$.

As shown by Peng [14], it is important to note that under sub-linear expectations the condition that “ \mathbf{Y} is independent to \mathbf{X} ” does not implies automatically that “ \mathbf{X} is independent to \mathbf{Y} ”.

From the definition of independence, it is easily seen that, if Y is independent to X and $X \geq 0, \widehat{\mathbb{E}}[Y] \geq 0$, then

$$\widehat{\mathbb{E}}[XY] = \widehat{\mathbb{E}}[X]\widehat{\mathbb{E}}[Y]. \quad (1.1)$$

Further, if Y is independent to X and $X \geq 0, Y \geq 0$, then

$$\widehat{\mathbb{E}}[XY] = \widehat{\mathbb{E}}[X]\widehat{\mathbb{E}}[Y], \quad \widehat{\mathcal{E}}[XY] = \widehat{\mathcal{E}}[X]\widehat{\mathcal{E}}[Y]. \quad (1.2)$$

If $\{X_n; n \geq 1\}$ is a sequence of independent random variables with both the upper-expectations $\widehat{\mathbb{E}}[X_i]$ and lower-expectations $\widehat{\mathcal{E}}[X_i]$ being zeros, then it is easily checked that

$$\widehat{\mathbb{E}}[S_n^2] = \widehat{\mathbb{E}}\left[\sum_{k=1}^n X_k^2 + \sum_{i \neq j} X_i X_j\right] = \sum_{k=1}^n \widehat{\mathbb{E}}[X_k^2],$$

because $\widehat{\mathbb{E}}[X_i X_j] = \widehat{\mathcal{E}}[X_i X_j] = 0$ for $i \neq j$ by the definition of the independence, where $S_n = \sum_{k=1}^n X_k$. However, when the popular truncation method is used for studying the limit theorems, the truncated random variables usually no longer have zero sub-linear expectations. It is hard to centralize a random variable such that its upper-expectation and lower-expectation are both zeros. But is easy to centralize a random variable X such that one of $\widehat{\mathbb{E}}[X]$ and $\widehat{\mathcal{E}}[X]$ is zero. For example, the random variable $X - \widehat{\mathbb{E}}[X]$ has zero upper-expectation. So, the moments of S_n with the condition $\widehat{\mathbb{E}}[X_i] = 0$ ($i = 1, \dots, n$) are much useful than those with the condition $\widehat{\mathbb{E}}[X_i] = \widehat{\mathcal{E}}[X_i] = 0$ ($i = 1, \dots, n$). Unfortunately, by noting that the independence of X and Y does not imply $\widehat{\mathbb{E}}[(X - \widehat{\mathbb{E}}[X])(Y - \widehat{\mathbb{E}}[Y])] = 0$ (or ≤ 0), even to get a good estimate of the second order moment $\widehat{\mathbb{E}}[(\sum_{k=1}^n (X_k - \widehat{\mathbb{E}}[X_k]))^2]$ is not a trivial work. As for the probability inequalities or moment inequalities for the maximum partial sums $\max_{k \leq n} S_k$, in the classical probability space, the proof depends basically on the additivity of the probabilities and the expectations. For example, the integral on the event $\{\max_{i \leq n} S_i \geq x\}$ is usually split to integrals on $\{\max_{i \leq k} S_i < x, S_k \geq x\}$, $k = 1, \dots, n$. The methods based on the additivity can not be used under the framework of sub-linear expectations. Other popular techniques such as the symmetrization, the martingale method and the stopping time method are also not available under the sub-linear expectations because they are essentially based on the additivity property. The main purpose of this paper is to establish the moment inequalities for $\max_{k \leq n} S_k$ which can be applied to truncated random variables freely. To explain our main idea, we first give the following result on Kolmogorov's inequality.

Theorem 1.4. (Kolmogorov's inequality) Let $\{X_1, \dots, X_n\}$ be a sequence of random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ with $\widehat{\mathbb{E}}[X_k] = 0$, $k = 1, \dots, n$. Suppose that X_k is independent to (X_{k+1}, \dots, X_n) for each $k = 1, \dots, n-1$. Denote $S_k = X_1 + \dots + X_k$, $S_0 = 0$. Then

$$\widehat{\mathbb{E}}[(\max_{k \leq n} S_k)^2] \leq \sum_{k=1}^n \widehat{\mathbb{E}}[X_k^2]. \quad (1.3)$$

In particular,

$$\widehat{\mathbb{E}}[(S_n^+)^2] \leq \sum_{k=1}^n \widehat{\mathbb{E}}[X_k^2].$$

Proof. Set $T_k = \max(X_k, X_k + X_{k+1}, \dots, X_k + \dots + X_n)$. Then $T_k, T_k^+ \in \mathcal{H}$, and $T_k = X_k + T_{k+1}^+$, $T_k^2 = X_k^2 + 2X_k T_{k+1}^+ + (T_{k+1}^+)^2$. It follows that

$$\widehat{\mathbb{E}}[T_k^2] \leq \widehat{\mathbb{E}}[X_k^2] + 2\widehat{\mathbb{E}}[X_k T_{k+1}^+] + \widehat{\mathbb{E}}[(T_{k+1}^+)^2].$$

Note $\widehat{\mathbb{E}}[X_k T_{k+1}^+] = 0$ by (1.1). We conclude that

$$\widehat{\mathbb{E}}[T_k^2] \leq \widehat{\mathbb{E}}[X_k^2] + \widehat{\mathbb{E}}[(T_{k+1}^+)^2] \leq \widehat{\mathbb{E}}[X_k^2] + \widehat{\mathbb{E}}[T_{k+1}^2].$$

Hence $\widehat{\mathbb{E}}[T_1^2] \leq \sum_{k=1}^n \widehat{\mathbb{E}}[X_k^2]$. The proof is completed. \square

In the above proof, the independence is utilized to get $\widehat{\mathbb{E}}[X_k T_{k+1}^+] \leq 0$ and so can be weakened. Recall that in the probability $(\Omega, \mathcal{F}, \mathbb{P})$, two random vectors $\mathbf{Y} = (Y_1, \dots, Y_n)$ and $\mathbf{X} = (X_1, \dots, X_m)$ are said to be negatively dependent if for each pair of coordinatewise nondecreasing (resp. non-increasing) functions $\varphi_1(\mathbf{x})$ and $\varphi_2(\mathbf{y})$ we have

$$E_{\mathbb{P}}[\varphi_1(\mathbf{X})\varphi_2(\mathbf{Y})] \leq E_{\mathbb{P}}[\varphi_1(\mathbf{X})]E_{\mathbb{P}}[\varphi_2(\mathbf{Y})]$$

whenever the expectations considered exist.

We introduce the concept of negative dependence under the sub-linear expectation.

Definition 1.5. (Negative dependence) In a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, a random vector $\mathbf{Y} = (Y_1, \dots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be negatively dependent to another random vector $\mathbf{X} = (X_1, \dots, X_m)$, $X_i \in \mathcal{H}$ under $\widehat{\mathbb{E}}$ if for each pair of test functions $\varphi_1 \in C_{l,Lip}(\mathbb{R}_m)$ and $\varphi_2 \in C_{l,Lip}(\mathbb{R}_n)$ we have

$$\widehat{\mathbb{E}}[\varphi_1(\mathbf{X})\varphi_2(\mathbf{Y})] \leq \widehat{\mathbb{E}}[\varphi_1(\mathbf{X})]\widehat{\mathbb{E}}[\varphi_2(\mathbf{Y})]$$

whenever $\varphi_1(\mathbf{X}) \geq 0$, $\widehat{\mathbb{E}}[\varphi_2(\mathbf{Y})] \geq 0$, $\widehat{\mathbb{E}}[|\varphi_1(\mathbf{X})\varphi_2(\mathbf{Y})|] < \infty$, $\widehat{\mathbb{E}}[|\varphi_1(\mathbf{X})|] < \infty$, $\widehat{\mathbb{E}}[|\varphi_2(\mathbf{Y})|] < \infty$, and either φ_1, φ_2 are coordinatewise nondecreasing or φ_1, φ_2 are coordinatewise non-increasing.

By the definition, it is easily seen that, if $\mathbf{Y} = (Y_1, \dots, Y_n)$ is negatively dependent to $\mathbf{X} = (X_1, \dots, X_m)$, $\varphi_1 \in C_{l,Lip}(\mathbb{R}_m)$ and $\varphi_2 \in C_{l,Lip}(\mathbb{R}_n)$ are coordinatewise nondecreasing (resp. non-increasing) functions, then $\varphi_2(\mathbf{Y})$ is negatively dependent to $\varphi_1(\mathbf{X})$. Further, if $Y \in \mathcal{H}$ is negatively dependent to $X \in \mathcal{H}$ and $X \geq 0$, $\widehat{\mathbb{E}}[X] < \infty$, $\widehat{\mathbb{E}}[|Y|] < \infty$, $\widehat{\mathbb{E}}[Y] \leq 0$, then

$$\widehat{\mathbb{E}}[YX] \leq \widehat{\mathbb{E}}[(Y - \widehat{\mathbb{E}}[Y])X] + \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[Y]X] \leq \widehat{\mathbb{E}}[Y - \widehat{\mathbb{E}}[Y]]\widehat{\mathbb{E}}[X] \leq 0.$$

It is obvious that, if \mathbf{Y} is independent to \mathbf{X} , then \mathbf{Y} is negatively dependent to \mathbf{X} . The following is the classical example introduced by Huber and Strassen [6].

Example 1.6. Let \mathcal{P} be a family of probability measures defined on (Ω, \mathcal{F}) . For any random variable ξ , we denote the upper expectation by

$$\widehat{\mathbb{E}}[\xi] = \sup_{Q \in \mathcal{P}} E_Q[\xi].$$

Then $\widehat{\mathbb{E}}[\cdot]$ is a sub-linear expectation. Moreover, if \mathbf{X} and \mathbf{Y} are independent under each $Q \in \mathcal{P}$, then \mathbf{Y} is negatively dependent to \mathbf{X} under $\widehat{\mathbb{E}}$. In fact,

$$\begin{aligned} \widehat{\mathbb{E}}[\varphi_1(\mathbf{X})\varphi_2(\mathbf{Y})] &= \sup_{Q \in \mathcal{P}} E_Q[\varphi_1(\mathbf{X})\varphi_2(\mathbf{Y})] = \sup_{Q \in \mathcal{P}} E_Q[\varphi_1(\mathbf{X})]E_Q[\varphi_2(\mathbf{Y})] \\ &\leq \sup_{Q \in \mathcal{P}} E_Q[\varphi_1(\mathbf{X})] \sup_{Q \in \mathcal{P}} E_Q[\varphi_2(\mathbf{Y})] = \widehat{\mathbb{E}}[\varphi_1(\mathbf{X})]\widehat{\mathbb{E}}[\varphi_2(\mathbf{Y})] \end{aligned}$$

whenever $\varphi_1(\mathbf{X}) \geq 0$ and $\widehat{\mathbb{E}}[\varphi_2(\mathbf{Y})] \geq 0$.

However, \mathbf{Y} may be not independent to \mathbf{X} .

With the similar argument, we can show that \mathbf{Y} is negatively dependent to \mathbf{X} under $\widehat{\mathbb{E}}$ if \mathbf{X} and \mathbf{Y} are negatively dependent under each $Q \in \mathcal{P}$.

According to its proof, the conclusion of Theorem 1.4 remains true under the concept of negative dependence.

Corollary 1.7. Let $\{X_1, \dots, X_n\}$ be a sequence of random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ with $\widehat{\mathbb{E}}[X_k] \leq 0$, $k = 1, \dots, n$. Suppose that X_k is negatively dependent to (X_{k+1}, \dots, X_n) for each $k = 1, \dots, n-1$. Then (1.3) holds.

Our basic idea for obtaining Theorem 1.4 comes from Newman and Wright [9] and Matula [8] where Kolmogorov's inequality is established for the classical positively and negatively dependent random variables respectively.

2 Rosenthal's inequalities

In this section, we extend Kolmogorov's inequality to Rosenthal's inequalities. For moment inequalities of partial sums of the classical negatively dependent random variables and related strong limit theorems, one can refer to Shao [18], Su, Zhao and Wang [17], Yuan and An [19], Zhang [20] [21] [22], Zhang and Wen [23] etc. Some techniques from these papers will be used in the lines of our proofs. We let $\{X_1, \dots, X_n\}$ be a sequence of random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, and denote $S_k = X_1 + \dots + X_k$, $S_0 = 0$.

Theorem 2.1. (Rosenthal's inequality) (a) Suppose that X_k is negatively dependent to (X_{k+1}, \dots, X_n) for each $k = 1, \dots, n-1$, and $\widehat{\mathbb{E}}[X_k] \leq 0$, $k = 1, \dots, n$. Then

$$\widehat{\mathbb{E}} \left[\left| \max_{k \leq n} S_k \right|^p \right] \leq 2^{2-p} \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p], \quad \text{for } 1 \leq p \leq 2 \quad (2.1)$$

and

$$\widehat{\mathbb{E}} \left[\left| \max_{k \leq n} S_k \right|^p \right] \leq C_p n^{p/2-1} \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p], \quad \text{for } p \geq 2. \quad (2.2)$$

(b) Suppose that X_k is independent to (X_{k+1}, \dots, X_n) for each $k = 1, \dots, n-1$, and $\widehat{\mathbb{E}}[X_k] \leq 0$, $k = 1, \dots, n$. Then

$$\widehat{\mathbb{E}} \left[\left| \max_{k \leq n} S_k \right|^p \right] \leq C_p \left\{ \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \left(\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^2] \right)^{p/2} \right\}, \quad \text{for } p \geq 2. \quad (2.3)$$

(c) In general, suppose that X_k is negatively dependent to (X_{k+1}, \dots, X_n) for each $k = 1, \dots, n-1$, or X_{k+1} is negatively dependent to (X_1, \dots, X_k) for each $k = 1, \dots, n-1$. Then

$$\begin{aligned} \widehat{\mathbb{E}} \left[\max_{k \leq n} |S_k|^p \right] &\leq C_p \left\{ \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \left(\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^2] \right)^{p/2} \right. \\ &\quad \left. + \left(\sum_{k=1}^n [(\widehat{\mathcal{E}}[X_k])^- + (\widehat{\mathbb{E}}[X_k])^+] \right)^p \right\}. \end{aligned} \quad (2.4)$$

Here C_p is a positive constant depending only on p .

If we consider the sequence $\{X_1, X_2, \dots, X_n\}$ in the reverse order as $\{X_n, X_{n-1}, \dots, X_1\}$, by Theorem 2.1 (a) and (b) we have the following corollary.

Corollary 2.2. Let $\{X_1, \dots, X_n\}$ be a sequence of random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ with $\widehat{\mathbb{E}}[X_k] \leq 0$, $k = 1, \dots, n$.

(a) Suppose that X_{k+1} is negatively dependent to (X_1, \dots, X_k) for each $k = 1, \dots, n-1$. Then

$$\widehat{\mathbb{E}} \left[\left| \max_{k \leq n} (S_n - S_k) \right|^p \right] \leq 2^{2-p} \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p], \quad \text{for } 1 \leq p \leq 2 \quad (2.5)$$

and

$$\widehat{\mathbb{E}} \left[\left| \max_{k \leq n} (S_n - S_k) \right|^p \right] \leq C_p n^{p/2-1} \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p], \quad \text{for } p \geq 2. \quad (2.6)$$

In particular,

$$\widehat{\mathbb{E}} \left[(S_n^+)^p \right] \leq \begin{cases} 2^{2-p} \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p], & \text{for } 1 \leq p \leq 2, \\ C_p n^{p/2-1} \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p], & \text{for } p \geq 2. \end{cases} \quad (2.7)$$

(b) Suppose that X_{k+1} is independent to (X_1, \dots, X_k) for each $k = 1, \dots, n-1$. Then

$$\widehat{\mathbb{E}} \left[\left| \max_{k \leq n} (S_n - S_k) \right|^p \right] \leq C_p \left\{ \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \left(\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^2] \right)^{p/2} \right\}, \quad \text{for } p \geq 2. \quad (2.8)$$

In particular,

$$\widehat{\mathbb{E}}[(S_n^+)^p] \leq C_p \left\{ \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \left(\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^2] \right)^{p/2} \right\}, \quad \text{for } p \geq 2.$$

For the moments under $\widehat{\mathcal{E}}$, we have the following estimates.

Theorem 2.3. Let $\{X_1, \dots, X_n\}$ be a sequence of random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ with $\widehat{\mathcal{E}}[X_k] \leq 0$, $k = 1, \dots, n$, and $1 \leq p \leq 2$. If X_k is independent to (X_{k+1}, \dots, X_n) for each $k = 1, \dots, n-1$, then

$$\widehat{\mathcal{E}} \left[\left| \max_{k \leq n} S_k \right|^p \right] \leq 2^{2-p} \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p], \quad \text{for } 1 \leq p \leq 2. \quad (2.9)$$

If X_{k+1} is independent to (X_1, \dots, X_k) for each $k = 1, \dots, n-1$, then

$$\widehat{\mathcal{E}} \left[\left| \max_{k \leq n} (S_n - S_k) \right|^p \right] \leq 2^{2-p} \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p], \quad \text{for } 1 \leq p \leq 2. \quad (2.10)$$

To prove Theorems 2.1-2.3, we need Hölder's inequality under the sub-linear expectation which can be proved by the same way under the linear expectation due to the properties of the monotonicity and sub-additivity (c.f. Proposition 1.16 of Peng [16]).

Lemma 2.4. (Hölder's inequality) Let $p, q > 1$ be two real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for two random variables X, Y in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ we have

$$\widehat{\mathbb{E}}[|XY|] \leq \left(\widehat{\mathbb{E}}[|X|^p] \right)^{\frac{1}{p}} \left(\widehat{\mathbb{E}}[|Y|^q] \right)^{\frac{1}{q}}.$$

Proof of Theorem 2.1. Let T_k be defined as in the proof of Theorem 1.4.

(a) We first prove (2.1). Substituting $x = X_k$ and $y = T_{k+1}^+$ to the following elementary inequality

$$|x + y|^p \leq 2^{2-p}|x|^p + |y|^p + px|y|^{p-1}\text{sgn}(y), \quad 1 \leq p \leq 2 \quad (2.11)$$

yields

$$\begin{aligned} \widehat{\mathbb{E}}[|T_k|^p] &\leq 2^{2-p}\widehat{\mathbb{E}}[|X_k|^p] + \widehat{\mathbb{E}}[(T_{k+1}^+)^p] + p\widehat{\mathbb{E}}[X_k(T_{k+1}^+)^{p-1}] \\ &\leq 2^{2-p}\widehat{\mathbb{E}}[|X_k|^p] + \widehat{\mathbb{E}}[|T_{k+1}|^p] \end{aligned}$$

by the definition of negative dependence and the facts that $\widehat{\mathbb{E}}[X_k] \leq 0$, $T_{k+1}^+ \geq 0$, and T_{k+1}^+ is a coordinatewise nondecreasing function of X_{k+1}, \dots, X_n . Hence

$$\widehat{\mathbb{E}}[|T_1|^p] \leq 2^{2-p} \sum_{k=1}^{n-1} \widehat{\mathbb{E}}[|X_k|^p] + \widehat{\mathbb{E}}[|X_n|^p].$$

So, (2.1) is proved.

For (2.2), by the following elementary inequality

$$|x + y|^p \leq 2^p p^2 |x|^p + |y|^p + px|y|^{p-1}\text{sgn}(y) + 2^p p^2 x^2 |y|^{p-2}, \quad p \geq 2,$$

we have

$$|T_k|^p \leq 2^p p^2 |X_k|^p + |T_{k+1}|^p + pX_k(T_{k+1}^+)^{p-1} + 2^p p^2 X_k^2 (T_{k+1}^+)^{p-2}.$$

It follows that

$$|T_i|^p \leq 2^p p^2 \sum_{k=i}^n |X_k|^p + p \sum_{k=i}^{n-1} X_k (T_{k+1}^+)^{p-1} + 2^p p^2 \sum_{k=i}^{n-1} X_k^2 (T_{k+1}^+)^{p-2}. \quad (2.12)$$

Hence by the definition of the negative dependence and Hölder's inequality,

$$\begin{aligned}\widehat{\mathbb{E}}[|T_i|^p] &\leq 2^p p^2 \widehat{\mathbb{E}} \left[\sum_{k=i}^n |X_k|^p \right] + p \sum_{k=i}^{n-1} \widehat{\mathbb{E}} [X_k (T_{k+1}^+)^{p-1}] + 2^p p^2 \sum_{k=i}^{n-1} \widehat{\mathbb{E}} [X_k^2 (T_{k+1}^+)^{p-2}] \\ &\leq 2^p p^2 \widehat{\mathbb{E}} \left[\sum_{k=1}^n |X_k|^p \right] + 2^p p^2 \sum_{k=1}^{n-1} \left(\widehat{\mathbb{E}}[|X_k|^p] \right)^{\frac{2}{p}} \left(\widehat{\mathbb{E}}[|T_{k+1}|^p] \right)^{1-\frac{2}{p}}.\end{aligned}$$

Let $A_n = \max_{k \leq n} \widehat{\mathbb{E}}[|T_k|^p]$. Then

$$A_n \leq 2^p p^2 \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + 2^p p^2 \sum_{k=1}^{n-1} \left(\widehat{\mathbb{E}}[|X_k|^p] \right)^{\frac{2}{p}} A_n^{1-\frac{2}{p}}.$$

From the above inequality, it can be shown that

$$A_n \leq C_p \left\{ \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \left(\sum_{k=1}^{n-1} \left(\widehat{\mathbb{E}}[|X_k|^p] \right)^{\frac{2}{p}} \right)^{\frac{p}{2}} \right\} \leq C_p n^{p/2-1} \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p].$$

(2.2) is proved.

(b) Note the independence. From (2.12) it follows that

$$\begin{aligned}\widehat{\mathbb{E}}[|T_i|^p] &\leq 2^p p^2 \widehat{\mathbb{E}} \left[\sum_{k=i}^n |X_k|^p \right] + p \sum_{k=i}^{n-1} \widehat{\mathbb{E}} [X_k (T_{k+1}^+)^{p-1}] + 2^p p^2 \sum_{k=i}^{n-1} \widehat{\mathbb{E}} [X_k^2 (T_{k+1}^+)^{p-2}] \\ &= 2^p p^2 \widehat{\mathbb{E}} \left[\sum_{k=i}^n |X_k|^p \right] + p \sum_{k=i}^{n-1} \widehat{\mathbb{E}}[X_k] \widehat{\mathbb{E}}[(T_{k+1}^+)^{p-1}] + 2^p p^2 \sum_{k=i}^{n-1} \widehat{\mathbb{E}}[X_k^2] \widehat{\mathbb{E}}[(T_{k+1}^+)^{p-2}] \\ &\leq 2^p p^2 \widehat{\mathbb{E}} \left[\sum_{k=1}^n |X_k|^p \right] + 2^p p^2 \sum_{k=1}^{n-1} \widehat{\mathbb{E}}[X_k^2] \left(\widehat{\mathbb{E}}[|T_{k+1}|^p] \right)^{1-\frac{2}{p}}.\end{aligned}$$

Let $A_n = \max_{k \leq n} \widehat{\mathbb{E}}[|T_k|^p]$. Then

$$A_n \leq 2^p p^2 \widehat{\mathbb{E}} \left[\sum_{k=1}^n |X_k|^p \right] + 2^p p^2 \sum_{k=1}^{n-1} \widehat{\mathbb{E}}[X_k^2] A_n^{1-\frac{2}{p}}.$$

From the above inequality, it can be shown that

$$A_n \leq C_p \left\{ \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \left(\sum_{k=1}^n \widehat{\mathbb{E}}[X_k^2] \right)^{\frac{p}{2}} \right\}.$$

(2.2) is proved. \square

(c) We first show the Marcinkiewicz-Zygmund inequality:

$$\widehat{\mathbb{E}}[\max_{k \leq n} |S_k|^p] \leq C_p \left\{ \left(\sum_{k=1}^n \left((\widehat{\mathbb{E}}[X_k])^+ + (\widehat{\mathbb{E}}[X_k])^- \right) \right)^p + \widehat{\mathbb{E}} \left(\sum_{k=1}^n X_k^2 \right)^{\frac{p}{2}} \right\}. \quad (2.13)$$

Without loss of generality, we assume that X_k is negatively dependent to (X_{k+1}, \dots, X_n) for all $k = 1, 2, \dots, n-1$. If X_{k+1} is negatively dependent to (X_1, \dots, X_k) for all $k = 1, 2, \dots, n-1$, then (2.13) will hold with $\max_{k \leq n} |S_k|$ being replaced by $\max_{0 \leq k \leq n} |S_n - S_k|$. By noting the fact $\max_{k \leq n} |S_k| \leq \max_{0 \leq k \leq n} |S_n - S_k| + |S_n| \leq 2 \max_{0 \leq k \leq n} |S_n - S_k|$, (2.13) also is true.

Write $\widetilde{T}_1 = \max_{k \leq n} |S_k|$. It is easily seen that $S_k + T_{k+1}^+ = \max(S_k, S_{k+1}, \dots, S_n) \leq T_1$. So, $T_{k+1}^+ \leq 2\widetilde{T}_1$. Note (2.12). By the definition of the negative dependence,

$$\widehat{\mathbb{E}}[X_k (T_{k+1}^+)^{p-1}] \leq \begin{cases} \widehat{\mathbb{E}}[X_k] \widehat{\mathbb{E}}[(T_{k+1}^+)^{p-1}], & \text{if } \widehat{\mathbb{E}}[X_k] \geq 0 \\ 0, & \text{if } \widehat{\mathbb{E}}[X_k] < 0 \end{cases}$$

$$\leq 2^{p-1} (\widehat{\mathbb{E}}[X_k])^+ \widehat{\mathbb{E}}[\widetilde{T}_1^{p-1}] \leq 2^{p-1} (\widehat{\mathbb{E}}[X_k])^+ (\widehat{\mathbb{E}}[\widetilde{T}_1^p])^{1-\frac{1}{p}}$$

by the Hölder inequality. By (2.12) and the Hölder inequality again, it follows that

$$\begin{aligned} \widehat{\mathbb{E}}[|T_1|^p] &\leq 2^p p^2 \widehat{\mathbb{E}} \left[\sum_{k=1}^n |X_k|^p \right] + p \sum_{k=1}^{n-1} \widehat{\mathbb{E}} [X_k (T_{k+1}^+)^{p-1}] + 2^p p^2 \widehat{\mathbb{E}} \left[\sum_{k=1}^{n-1} X_k^2 (T_{k+1}^+)^{p-2} \right] \\ &\leq 2^p p^2 \widehat{\mathbb{E}} \left[\sum_{k=1}^n |X_k|^p \right] + 2^{p-1} p \sum_{k=1}^{n-1} (\widehat{\mathbb{E}}[X_k])^+ (\widehat{\mathbb{E}}[\widetilde{T}_1^p])^{1-\frac{1}{p}} \\ &\quad + 2^p p^2 2^{p-2} \widehat{\mathbb{E}} \left[\sum_{k=1}^{n-1} X_k^2 \widetilde{T}_1^{p-2} \right] \\ &\leq 2^p p^2 \widehat{\mathbb{E}} \left[\sum_{k=1}^n |X_k|^p \right] + 2^{p-1} p \left(\sum_{k=1}^{n-1} (\widehat{\mathbb{E}}[X_k])^+ \right) (\widehat{\mathbb{E}}[\widetilde{T}_1^p])^{1-\frac{1}{p}} \\ &\quad + 2^{2p-2} p^2 \left[\widehat{\mathbb{E}} \left(\sum_{k=1}^{n-1} X_k^2 \right)^{\frac{p}{2}} \right]^{\frac{2}{p}} (\widehat{\mathbb{E}}[\widetilde{T}_1^p])^{1-\frac{2}{p}}. \end{aligned}$$

Similarly,

$$\begin{aligned} \widehat{\mathbb{E}} \left[\left| \max_{k \leq n} (-S_k) \right|^p \right] &\leq 2^p p^2 \widehat{\mathbb{E}} \left[\sum_{k=1}^n |X_k|^p \right] + 2^{p-1} p \left(\sum_{k=1}^{n-1} (\widehat{\mathbb{E}}[-X_k])^+ \right) (\widehat{\mathbb{E}}[\widetilde{T}_1^p])^{1-\frac{1}{p}} \\ &\quad + 2^{2p-2} p^2 \left[\widehat{\mathbb{E}} \left(\sum_{k=1}^{n-1} X_k^2 \right)^{\frac{p}{2}} \right]^{\frac{2}{p}} (\widehat{\mathbb{E}}[\widetilde{T}_1^p])^{1-\frac{2}{p}}. \end{aligned}$$

Hence

$$\begin{aligned} \widehat{\mathbb{E}}[\widetilde{T}_1^p] &\leq 2^{p+1} p^2 \widehat{\mathbb{E}} \left[\sum_{k=1}^n |X_k|^p \right] + 2^{p-1} p \left(\sum_{k=1}^n [(\widehat{\mathbb{E}}[X_k])^+ + (\widehat{\mathbb{E}}[X_k])^-] \right) (\widehat{\mathbb{E}}[\widetilde{T}_1^p])^{1-\frac{1}{p}} \\ &\quad + 2^{2p-1} p^2 \left[\widehat{\mathbb{E}} \left(\sum_{k=1}^{n-1} X_k^2 \right)^{\frac{p}{2}} \right]^{\frac{2}{p}} (\widehat{\mathbb{E}}[\widetilde{T}_1^p])^{1-\frac{2}{p}}, \end{aligned}$$

which implies

$$\widehat{\mathbb{E}}[\widetilde{T}_1^p] \leq C_p \left\{ \widehat{\mathbb{E}} \left[\sum_{k=1}^n |X_k|^p \right] + \left(\sum_{k=1}^n ((\widehat{\mathbb{E}}[X_k])^+ + (\widehat{\mathbb{E}}[X_k])^-) \right)^p + \widehat{\mathbb{E}} \left(\sum_{k=1}^n X_k^2 \right)^{\frac{p}{2}} \right\}.$$

Note

$$\left(\sum_{k=1}^n |x_k|^p \right)^{\frac{2}{p}} = \left(\sum_{k=1}^n |x_k^2|^{\frac{p}{2}} \right)^{\frac{2}{p}} \leq \sum_{k=1}^n x_k^2 \quad \text{for } \frac{2}{p} \leq 1.$$

So

$$\sum_{k=1}^n |X_k|^p \leq \left(\sum_{k=1}^n X_k^2 \right)^{\frac{p}{2}}.$$

The Marcinkiewicz-Zygmund inequality (2.13) is proved.

Now, for $2 \leq p \leq 4$, applying (2.1) to the sequences $\{(X_1^+)^2, \dots, (X_n^+)^2\}$ yields

$$\widehat{\mathbb{E}} \left[\left(\left\{ \sum_{k=1}^n [(X_k^+)^2 - \widehat{\mathbb{E}}[(X_k^+)^2]] \right\}^+ \right)^{\frac{p}{2}} \right]$$

$$\leq 2^{2-\frac{p}{2}} \sum_{k=1}^n \widehat{\mathbb{E}}[|(X_k^+)^2 - \widehat{\mathbb{E}}[(X_k^+)^2]|^{\frac{p}{2}}] \leq C_p \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p].$$

It follows that

$$\widehat{\mathbb{E}}\left(\sum_{k=1}^n (X_k^+)^2\right)^{\frac{p}{2}} \leq C_p \left\{ \left(\sum_{k=1}^n \widehat{\mathbb{E}}[(X_k^+)^2]\right)^{\frac{p}{2}} + \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] \right\}.$$

Similarly

$$\widehat{\mathbb{E}}\left(\sum_{k=1}^n (X_k^-)^2\right)^{\frac{p}{2}} \leq C_p \left\{ \left(\sum_{k=1}^n \widehat{\mathbb{E}}[(X_k^-)^2]\right)^{\frac{p}{2}} + \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] \right\}.$$

Hence

$$\widehat{\mathbb{E}}\left(\sum_{k=1}^n X_k^2\right)^{\frac{p}{2}} \leq C_p \left\{ \left(\sum_{k=1}^n \widehat{\mathbb{E}}[X_k^2]\right)^{\frac{p}{2}} + \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] \right\}.$$

Substituting the above estimate to (2.13) yield (2.3).

Suppose (2.3) is proved for $2^l < p \leq 2^{l+1}$. Then applying it to the sequences $\{(X_1^+)^2, \dots, (X_n^+)^2\}$ and $\{(X_1^-)^2, \dots, (X_n^-)^2\}$ respectively with $2^l < p/2 \leq 2^{l+1}$ yields

$$\begin{aligned} \widehat{\mathbb{E}}\left[\left(\sum_{k=1}^n (X_k^+)^2\right)^{\frac{p}{2}}\right] &\leq C_p \left\{ \sum_{k=1}^n \widehat{\mathbb{E}}[|(X_k^+)^2|^{\frac{p}{2}}] + \left(\sum_{k=1}^n (\widehat{\mathbb{E}}[(X_k^+)^2])^{\frac{p}{2}} + \left(\sum_{k=1}^n \widehat{\mathbb{E}}[|(X_k^+)^2|^2]\right)^{\frac{p}{4}} \right\} \\ &\leq C_p \left\{ \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \left(\sum_{k=1}^n \widehat{\mathbb{E}}[X_k^2]\right)^{\frac{p}{2}} + \left(\sum_{k=1}^n \widehat{\mathbb{E}}[X_k^4]\right)^{\frac{p}{4}} \right\} \end{aligned}$$

and

$$\widehat{\mathbb{E}}\left[\left(\sum_{k=1}^n (X_k^-)^2\right)^{\frac{p}{2}}\right] \leq C_p \left\{ \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \left(\sum_{k=1}^n \widehat{\mathbb{E}}[X_k^2]\right)^{\frac{p}{2}} + \left(\sum_{k=1}^n \widehat{\mathbb{E}}[X_k^4]\right)^{\frac{p}{4}} \right\}.$$

Hence

$$\widehat{\mathbb{E}}\left[\left(\sum_{k=1}^n X_k^2\right)^{\frac{p}{2}}\right] \leq C_p \left\{ \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \left(\sum_{k=1}^n \widehat{\mathbb{E}}[X_k^2]\right)^{\frac{p}{2}} + \left(\sum_{k=1}^n \widehat{\mathbb{E}}[X_k^4]\right)^{\frac{p}{4}} \right\}. \quad (2.14)$$

By applying Hölder's inequality, it follows that

$$\widehat{\mathbb{E}}[X_k^4] = \widehat{\mathbb{E}}\left[\left(X_k^2\right)^{\frac{p-4}{p-2}} \left(|X_k|^p\right)^{\frac{2}{p-2}}\right] \leq \left(\widehat{\mathbb{E}}[X_k^2]\right)^{\frac{p-4}{p-2}} \left(\widehat{\mathbb{E}}[|X_k|^p]\right)^{\frac{2}{p-2}},$$

which implies

$$\left(\sum_{k=1}^n \widehat{\mathbb{E}}[X_k^4]\right)^{p/4} \leq C_p \left\{ \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \left(\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^2]\right)^{p/2} \right\} \quad (2.15)$$

by some elementary calculation. Substituting (2.14) and (2.15) to (2.13), we conclude that (2.3) is also valid for $2^{l+1} < p \leq 2^{l+2}$. By the induction, (2.3) proved. \square

Proof of Theorem 2.3. Suppose that X_k is independent to (X_{k+1}, \dots, X_n) for each $k = 1, \dots, n-1$. Due to (2.11), we have

$$|T_k|^p \leq 2^{2-p} |X_k|^p + (T_{k+1}^+)^p + p X_k (T_{k+1}^+)^{p-1}.$$

By the independence and the fact that $\widehat{\mathcal{E}}[X + Y] \leq \widehat{\mathcal{E}}[X] + \widehat{\mathbb{E}}[Y]$, it follows that

$$\begin{aligned} & \widehat{\mathcal{E}}[2^{2-p}|X_k|^p + (T_{k+1}^+)^p + pX_k(T_{k+1}^+)^{p-1}|T_{k+1}^+] \\ & \leq 2^{2-p}\widehat{\mathbb{E}}[|X_k|^p] + (T_{k+1}^+)^p + p\widehat{\mathcal{E}}[X_k](T_{k+1}^+)^{p-1} \leq 2^{2-p}\widehat{\mathbb{E}}[|X_k|^p] + (T_{k+1}^+)^p. \end{aligned}$$

So

$$\widehat{\mathcal{E}}[|T_k|^p] \leq \widehat{\mathcal{E}}[2^{2-p}|X_k|^p + (T_{k+1}^+)^p + pX_k(T_{k+1}^+)^{p-1}] \leq 2^{2-p}\widehat{\mathbb{E}}[|X_k|^p] + \widehat{\mathcal{E}}[|T_{k+1}|^p].$$

It follows that

$$\widehat{\mathcal{E}}[|T_1|^p] \leq 2^{2-p} \sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p].$$

Now, (2.9) is proved. (2.10) follows from (2.9) by considering the sequence $\{X_1, X_2, \dots, X_n\}$ in the reverse order as $\{X_n, X_{n-1}, \dots, X_1\}$. \square

3 Strong laws of large numbers under capacities

Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \rightarrow [0, 1]$ is called a capacity if

$$V(\emptyset) = 0, V(\Omega) = 1 \text{ and } V(A) \leq V(B) \text{ } \forall A \subset B, A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$.

Here we only consider the capacities generated by a sub-linear expectation. Let $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ be a sub-linear space, and $\widehat{\mathcal{E}}$ be the conjugate expectation of $\widehat{\mathbb{E}}$. Furthermore, let us denote a pair (\mathbb{V}, \mathbb{V}) of capacities by

$$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\}, \quad \mathbb{V}(A) := 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F},$$

where A^c is the complement set of A . Then

$$\begin{aligned} \mathbb{V}(A) &:= \widehat{\mathbb{E}}[I_A], \quad \mathbb{V}(A) := \widehat{\mathcal{E}}[I_A], \quad \text{if } I_A \in \mathcal{H} \\ \widehat{\mathbb{E}}[f] &\leq \mathbb{V}(A) \leq \widehat{\mathbb{E}}[g], \quad \widehat{\mathcal{E}}[f] \leq \mathbb{V}(A) \leq \widehat{\mathcal{E}}[g], \quad \text{if } f \leq I_A \leq g, f, g \in \mathcal{H}. \end{aligned} \tag{3.1}$$

The corresponding Choquet integrals/expecations $(C_{\mathbb{V}}, C_{\mathbb{V}})$ are defined by

$$C_{\mathbb{V}}[X] = \int_0^\infty V(X \geq t)dt + \int_{-\infty}^0 [V(X \geq t) - 1]dt$$

with V being replaced by \mathbb{V} and \mathbb{V} respectively.

Definition 3.1. (I) A sub-linear expectation $\widehat{\mathbb{E}} : \mathcal{H} \rightarrow \mathbb{R}$ is called to be countably sub-additive if it satisfies

- (e) **Countable sub-additivity:** $\widehat{\mathbb{E}}[X] \leq \sum_{n=1}^\infty \widehat{\mathbb{E}}[X_n]$, whenever $X \leq \sum_{n=1}^\infty X_n$, $X, X_n \in \mathcal{H}$ and $X \geq 0, X_n \geq 0, n = 1, 2, \dots$;

It is called continuous if it satisfies

- (f) **Continuity from below:** $\widehat{\mathbb{E}}[X_n] \uparrow \widehat{\mathbb{E}}[X]$ if $0 \leq X_n \uparrow X$, where $X_n, X \in \mathcal{H}$;
 (g) **Continuity from above:** $\widehat{\mathbb{E}}[X_n] \downarrow \widehat{\mathbb{E}}[X]$ if $0 \leq X_n \downarrow X$, where $X_n, X \in \mathcal{H}$.

(II) A function $V : \mathcal{F} \rightarrow [0, 1]$ is called to be countably sub-additive if

$$V\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty V(A_n) \quad \forall A_n \in \mathcal{F}.$$

(III) A capacity $V : \mathcal{F} \rightarrow [0, 1]$ is called a continuous capacity if it satisfies

(III1) **Ccontinuity from below:** $V(A_n) \uparrow V(A)$ if $A_n \uparrow A$, where $A_n, A \in \mathcal{F}$;

(III2) **Continuity from above:** $V(A_n) \downarrow V(A)$ if $A_n \downarrow A$, where $A_n, A \in \mathcal{F}$.

Example 1.6 (continued) The sub-linear expectation $\hat{\mathbb{E}}$ defined in Example 1.6 is continuous from below, and so is countably sub-additive. If \mathcal{H} is the set of all random variables and \mathcal{P} is a weakly compact set of probability measures defined on (Ω, \mathcal{F}) , then $(\mathbb{V}, \mathcal{V})$ is a pair of continuous capacities.

Definition 3.2. Let $\{X_n; n \geq 1\}$ be a sequence of random variables in the sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. X_1, X_2, \dots are said to be independent if X_{i+1} is independent to (X_1, \dots, X_i) for each $i \geq 1$, they are said to be negatively dependent if X_{i+1} is negatively dependent to (X_1, \dots, X_i) for each $i \geq 1$, and they are said to be identically distributed if $X_i \stackrel{d}{=} X_1$ for each $i \geq 1$.

It is obvious that, if $\{X_n; n \geq 1\}$ is a sequence of independent random variables and $f_1(x), f_2(x), \dots \in C_{l,Lip}(\mathbb{R})$, then $\{f_n(X_n); n \geq 1\}$ is also a sequence of independent random variables; if $\{X_n; n \geq 1\}$ is a sequence of negatively dependent random variables and $f_1(x), f_2(x), \dots \in C_{l,Lip}(\mathbb{R})$ are non-decreasing (resp. non-increasing) functions, then $\{f_n(X_n); n \geq 1\}$ is also a sequence of negatively dependent random variables.

For a sequence $\{X_n; n \geq 1\}$ of random variables in the sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, we denote $S_n = \sum_{k=1}^n X_k$, $S_0 = 0$. The main purpose of this section is to establish the following Kolmogorov type strong laws of larger numbers.

Theorem 3.3. (a) Let $\{X_n; n \geq 1\}$ be a sequence of negatively dependent and identically distributed random variables. Suppose that \mathbb{V} is countably sub-additive, $C_V[|X_1|] < \infty$ and $\lim_{c \rightarrow \infty} \hat{\mathbb{E}}[(|X_1| - c)^+] = 0$. Then

$$\mathbb{V} \left(\left\{ \liminf_{n \rightarrow \infty} \frac{S_n}{n} < \hat{\mathbb{E}}[X_1] \right\} \cup \left\{ \limsup_{n \rightarrow \infty} \frac{S_n}{n} > \hat{\mathbb{E}}[X_1] \right\} \right) = 0. \quad (3.2)$$

(b) Suppose that $\{X_n; n \geq 1\}$ is a sequence of independent and identically distributed random variables, and \mathbb{V} is continuous. If

$$\mathbb{V} \left(\limsup_{n \rightarrow \infty} \frac{|S_n|}{n} = +\infty \right) < 1, \quad (3.3)$$

then $C_V[|X_1|] < \infty$.

(c) Suppose that $\{X_n; n \geq 1\}$ is a sequence of independent and identically distributed random variables with $C_V[|X_1|] < \infty$ and $\lim_{c \rightarrow \infty} \hat{\mathbb{E}}[(|X_1| - c)^+] = 0$. If \mathbb{V} is continuous, then

$$\mathbb{V} \left(\liminf_{n \rightarrow \infty} \frac{S_n}{n} = \hat{\mathbb{E}}[X_1] \text{ and } \limsup_{n \rightarrow \infty} \frac{S_n}{n} = \hat{\mathbb{E}}[X_1] \right) = 1 \quad (3.4)$$

and

$$\mathbb{V} \left(C \left\{ \frac{S_n}{n} \right\} = [\hat{\mathbb{E}}[X_1], \hat{\mathbb{E}}[X_1]] \right) = 1, \quad (3.5)$$

where $C(\{x_n\})$ denotes the cluster set of a sequence of $\{x_n\}$ in \mathbb{R} .

The following corollary follows from Theorem 3.3 immediately.

Corollary 3.4. Suppose that \mathcal{H} is a monotone class in the sense that $X \in \mathcal{H}$ whenever $\mathcal{H} \ni X_n \downarrow X \geq 0$. Assume that $\hat{\mathbb{E}}$ is continuous. Let $\{X_n; n \geq 1\}$ be a sequence of independent and identically distributed random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. Then

$$(3.3) \implies C_V[|X_1|] < \infty \implies (3.2).$$

Because \mathbb{V} may be not countably sub-additive in general, we define an outer capacity \mathbb{V}^* by

$$\mathbb{V}^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \mathbb{V}(A_n) : A \subset \bigcup_{n=1}^{\infty} A_n \right\}, \quad \mathcal{V}^*(A) = 1 - \mathbb{V}^*(A^c), \quad A \in \mathcal{F}.$$

Then it can be shown that $\mathbb{V}^*(A)$ is a countably sub-additive capacity with $\mathbb{V}^*(A) \leq \mathbb{V}(A)$ and the following properties:

(a*) If \mathbb{V} is countably sub-additive, then $\mathbb{V}^* \equiv \mathbb{V}$.

(b*) If $I_A \leq g$, $g \in \mathcal{H}$, then $\mathbb{V}^*(A) \leq \widehat{\mathbb{E}}[g]$. Further, if $\widehat{\mathbb{E}}$ is countably sub-additive, then

$$\widehat{\mathbb{E}}[f] \leq \mathbb{V}^*(A) \leq \mathbb{V}(A) \leq \widehat{\mathbb{E}}[g], \quad \forall f \leq I_A \leq g, f, g \in \mathcal{H}. \quad (3.6)$$

(c*) \mathbb{V}^* is the largest countably sub-additive capacity satisfying the property that $\mathbb{V}^*(A) \leq \widehat{\mathbb{E}}[g]$ whenever $I_A \leq g \in \mathcal{H}$, i.e., if V is also a countably sub-additive capacity satisfying $V(A) \leq \widehat{\mathbb{E}}[g]$ whenever $I_A \leq g \in \mathcal{H}$, then $V(A) \leq \mathbb{V}^*(A)$.

In fact, it is obvious that (c*) implies (a*). For (b*) and (c*), suppose $A \subset \bigcup_{n=1}^{\infty} A_n$, $\sum_{n=1}^{\infty} \mathbb{V}(A_n) \leq \mathbb{V}^*(A) + \epsilon/2$ with $I_{A_n} \leq f_n \in \mathcal{H}$ and $\widehat{\mathbb{E}}[f_n] \leq \mathbb{V}(A_n) + \epsilon/2^{n+2}$. If $\mathcal{H} \ni f \leq I_A$, then $f \leq \sum_{n=1}^{\infty} I_{A_n} \leq \sum_{n=1}^{\infty} f_n$, which implies

$$\widehat{\mathbb{E}}[f] \leq \sum_{n=1}^{\infty} \widehat{\mathbb{E}}[f_n] \leq \sum_{n=1}^{\infty} \mathbb{V}(A_n) + \sum_{n=1}^{\infty} \epsilon/2^{n+2} \leq \mathbb{V}^*(A) + \epsilon$$

by the countable sub-additivity of $\widehat{\mathbb{E}}$. While, if V is countably sub-additive, then

$$V(A) \leq \sum_{n=1}^{\infty} V(A_n) \leq \sum_{n=1}^{\infty} \widehat{\mathbb{E}}[f_n] \leq \sum_{n=1}^{\infty} \mathbb{V}(A_n) + \sum_{n=1}^{\infty} \epsilon/2^{n+2} \leq \mathbb{V}^*(A) + \epsilon.$$

Theorem 3.5. Let $\{X_n; n \geq 1\}$ be a sequence identically distributed random variables in $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$.

(a) Suppose that X_1, X_2, \dots are negatively dependent with $C_V[|X_1|] < \infty$ and $\lim_{c \rightarrow \infty} \widehat{\mathbb{E}}[(|X_1| - c)^+] = 0$. Then

$$\mathbb{V}^* \left(\left\{ \liminf_{n \rightarrow \infty} \frac{S_n}{n} < \widehat{\mathcal{E}}[X_1] \right\} \cup \left\{ \limsup_{n \rightarrow \infty} \frac{S_n}{n} > \widehat{\mathbb{E}}[X_1] \right\} \right) = 0. \quad (3.7)$$

(b) Suppose that X_1, X_2, \dots are independent, \mathbb{V}^* is continuous and $\widehat{\mathbb{E}}$ is countably sub-additive. If

$$\mathbb{V}^* \left(\limsup_{n \rightarrow \infty} \frac{|S_n|}{n} = +\infty \right) < 1, \quad (3.8)$$

then $C_V[|X_1|] < \infty$.

For proving the theorems, we need some properties of the sub-linear expectations and capacities. We define an extension of $\widehat{\mathbb{E}}$ on the space of all random variables by

$$\mathbb{E}^*[X] = \inf \{ \widehat{\mathbb{E}}[Y] : X \leq Y, Y \in \mathcal{H} \}.$$

Then \mathbb{E}^* is a sub-linear expectation on the space of all random variables, and

$$\mathbb{E}^*[X] = \widehat{\mathbb{E}}[X] \quad \forall X \in \mathcal{H}, \quad \mathbb{V}(A) = \mathbb{E}^*[I_A] \quad \forall A \in \mathcal{F}.$$

We have the following properties.

Lemma 3.6. (P1) If $\widehat{\mathbb{E}}$ is continuous from below, then it is countably sub-additive; Similarly, if \mathbb{V} is continuous from below, then it is countably sub-additive;

(P2) If \mathbb{V} is continuous from above, then \mathbb{V} and \mathcal{V} are continuous;

(P3) If $\widehat{\mathbb{E}}$ is continuous from above, then $\widehat{\mathbb{E}}$ is continuous from below controlled, that is, $\widehat{\mathbb{E}}[X_n] \uparrow \widehat{\mathbb{E}}[X]$ if $0 \leq X_n \uparrow X$, where $X_n, X \in \mathcal{H}$ and $\widehat{\mathbb{E}}X < \infty$;

(P4) Suppose that $\widehat{\mathbb{E}}$ is countably sub-additive. If $X \leq \sum_{n=1}^{\infty} X_n$, $X, X_n \geq 0$ and $X \in \mathcal{H}$, then $\widehat{\mathbb{E}}[X] \leq \sum_{n=1}^{\infty} \mathbb{E}^*[X_n]$;

(P5) Set $\mathcal{H} = \{A : I_A \in \mathcal{H}\}$, then \mathbb{V} is a countably sub-additive capacity in \mathcal{H} if $\widehat{\mathbb{E}}$ is countably sub-additive in \mathcal{H} , and, $(\mathbb{V}, \mathcal{V})$ is a pair of continuous capacities in \mathcal{H} if $\widehat{\mathbb{E}}$ is continuous in \mathcal{H} .

Proof. . For (P1), if $0 \leq X \leq \sum_{k=1}^{\infty} X_k$, $0 \leq X, X_n \in \mathcal{H}$, then

$$\begin{aligned} \widehat{\mathbb{E}}[X] &= \widehat{\mathbb{E}} \left[\left(\sum_{k=1}^{\infty} X_k \right) \wedge X \right] = \lim_{n \rightarrow \infty} \widehat{\mathbb{E}} \left[\left(\sum_{k=1}^n X_k \right) \wedge X \right] \\ &\leq \lim_{n \rightarrow \infty} \widehat{\mathbb{E}} \left[\sum_{k=1}^n X_k \right] \leq \lim_{n \rightarrow \infty} \sum_{k=1}^n \widehat{\mathbb{E}}[X_k] \leq \sum_{k=1}^{\infty} \widehat{\mathbb{E}}[X_k]. \end{aligned}$$

(P1) is proved.

For (P2), it is sufficient to note that, if $A_n \uparrow A$, then $A \setminus A_n \downarrow \emptyset$ and $0 \leq \mathbb{V}(A) - \mathbb{V}(A_n) \leq \mathbb{V}(A \setminus A_n)$. Similarly, for (P3), it is sufficient to note that $X - X_n \downarrow 0$ and $0 \leq \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[X_n] \leq \widehat{\mathbb{E}}[X - X_n]$.

For (P4), choose $0 \leq Y_n \in \mathcal{H}$ such that $Y_n \geq X_n$, $\widehat{\mathbb{E}}[Y_n] \leq \mathbb{E}^*[X_n] + \frac{\epsilon}{2^{n+1}}$. Then $X \leq \sum_{n=1}^{\infty} Y_n$. By the countable sub-additivity of $\widehat{\mathbb{E}}$,

$$\widehat{\mathbb{E}}[X] \leq \sum_{n=1}^{\infty} \widehat{\mathbb{E}}[Y_n] \leq \sum_{n=1}^{\infty} (\mathbb{E}^*[X_n] + \frac{\epsilon}{2^{n+1}}) \leq \sum_{n=1}^{\infty} \mathbb{E}^*[X_n] + \epsilon.$$

(P4) is proved. (P5) is obvious. \square

The following is the “the convergence part” of the Borel-Cantelli Lemma for a countably sub-additive capacity.

Lemma 3.7. (Borel-Cantelli’s Lemma) Let $\{A_n, n \geq 1\}$ be a sequence of events in \mathcal{F} . Suppose that V is a countably sub-additive capacity. If $\sum_{n=1}^{\infty} V(A_n) < \infty$, then

$$V(A_n \text{ i.o.}) = 0, \quad \text{where } \{A_n \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i.$$

Proof. By the monotonicity and countable sub-additivity, it follows that

$$0 \leq V \left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i \right) \leq V \left(\bigcup_{i=n}^{\infty} A_i \right) \leq \sum_{i=n}^{\infty} V(A_i) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

\square

Remark 3.8. It is important to note that the condition that “ X is independent to Y under $\widehat{\mathbb{E}}$ ” does not implies that “ X is independent to Y under \mathbb{V} ” because the indicator functions $I\{X \in A\}$ and $I\{Y \in A\}$ are not in $C_{l,Lip}(\mathbb{R})$, and also, “ X is independent to Y under \mathbb{V} ” does not implies that “ X is independent to Y under $\widehat{\mathbb{E}}$ ” because $\widehat{\mathbb{E}}$ is not an integral with respect to \mathbb{V} . So, we have not “the divergence part” of the Borel-Cantelli Lemma.

Similarly, the conditions that “ X and Y are identically distributed under $\widehat{\mathbb{E}}$ ” and that that “ X and Y are identically distributed under \mathbb{V} ” do not implies each other.

Lemma 3.9. Suppose that $X \in \mathcal{H}$ and $C_{\mathbb{V}}(|X|) < \infty$.

(a) Then

$$\sum_{j=1}^{\infty} \frac{\widehat{\mathbb{E}}[(|X| \wedge j)^2]}{j^2} < \infty. \quad (3.9)$$

(b) Furthermore, if $\lim_{c \rightarrow \infty} \widehat{\mathbb{E}}[|X| \wedge c] = \widehat{\mathbb{E}}[|X|]$, then

$$\widehat{\mathbb{E}}[|X|] \leq C_{\mathbb{V}}(|X|). \quad (3.10)$$

(c) If $\widehat{\mathbb{E}}$ is countably sub-additive, then

$$\widehat{\mathbb{E}}[|Y|] \leq C_V(|Y|), \quad \forall Y \in \mathcal{H} \quad (3.11)$$

and

$$\lim_{c \rightarrow \infty} \widehat{\mathbb{E}}[(|X| - c)^+] = 0, \quad \lim_{c \rightarrow \infty} \widehat{\mathbb{E}}[|X| \wedge c] = \widehat{\mathbb{E}}[|X|] \quad (3.12)$$

whenever $C_V(|X|) < \infty$.

Proof. (a) Note

$$\begin{aligned} (|X| \wedge j)^2 &= \sum_{i=1}^j |X|^2 I\{i-1 < |X| \leq i\} + j^2 I\{|X| > j\} \\ &\leq \sum_{i=1}^j i^2 I\{i-1 < |X| \leq i\} + j^2 I\{|X| > j\} \\ &= \sum_{i=0}^{j-1} (i+1)^2 I\{|X| > i\} - \sum_{i=1}^j i^2 I\{|X| > i\} + j^2 I\{|X| > j\} \\ &\leq 1 + \sum_{i=1}^{j-1} (2i+1) I\{|X| > i\} \\ &\leq 1 + 3 \sum_{i=1}^j i I\{|X| > i\}. \end{aligned}$$

So,

$$\widehat{\mathbb{E}}[(|X| \wedge j)^2] = \mathbb{E}^*[(|X| \wedge j)^2] \leq 1 + 3 \sum_{i=1}^j i \mathbb{V}(|X| > i),$$

by the (finite) sub-additivity of \mathbb{E}^* . It follows that

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{\widehat{\mathbb{E}}[(|X| \wedge j)^2]}{j^2} &\leq \sum_{j=1}^{\infty} \frac{1 + 3 \sum_{i=1}^j i \mathbb{V}(|X| > i)}{j^2} \\ &\leq 2 + 3 \sum_{i=1}^{\infty} i \mathbb{V}(|X| > i) \sum_{j=i+1}^{\infty} \frac{1}{j^2} \leq 2 + 3 \sum_{i=1}^{\infty} \mathbb{V}(|X| > i) \leq 2 + 3C_V(|X|). \end{aligned}$$

(3.9) is proved.

(b) For $n > 2$, note

$$\begin{aligned} |X| \wedge n &= \sum_{i=1}^n |X| I\{i-1 < |X| \leq i\} + n I\{|X| > n\} \\ &\leq \sum_{i=1}^n i (I\{|X| > i-1\} - I\{|X| > i\}) + n I\{|X| > n\} \leq 1 + \sum_{i=1}^n I\{|X| > i\}. \end{aligned}$$

It follows that

$$\widehat{\mathbb{E}}[|X| \wedge n] = \mathbb{E}^*[|X| \wedge n] \leq 1 + \sum_{i=1}^n \mathbb{V}(|X| \geq i) \leq 1 + \int_0^n \mathbb{V}(|X| \geq x) dx.$$

Taking $n \rightarrow \infty$ yields

$$\widehat{\mathbb{E}}[|X|] = \lim_{n \rightarrow \infty} \widehat{\mathbb{E}}[|X| \wedge n] \leq 1 + C_V(|X|).$$

By considering $|X|/\epsilon$ instead of $|X|$, we have

$$\widehat{\mathbb{E}}\left[\frac{|X|}{\epsilon}\right] \leq 1 + C_V\left(\frac{|X|}{\epsilon}\right) = 1 + \frac{1}{\epsilon} C_V(|X|).$$

That is $\widehat{\mathbb{E}}[|X|] \leq \epsilon + C_V(|X|)$. Taking $\epsilon \rightarrow 0$ yields (3.10).

(c) Now, from the fact that $|Y| \leq 1 + \sum_{i=1}^{\infty} I\{|Y| \geq i\}$, by the countable sub-additivity of $\widehat{\mathbb{E}}$ and Property (P4) in Lemma 3.6, it follows that

$$\widehat{\mathbb{E}}[|Y|] \leq 1 + \sum_{i=1}^{\infty} \mathbb{E}^*[I\{|Y| \geq i\}] = 1 + \sum_{i=1}^{\infty} \mathbb{V}(|Y| \geq i) \leq 1 + C_V(|Y|).$$

And then (3.11) is proved by the same argument in (b) above.

Letting $Y = (X - c)^+$ in (3.11) yields

$$\widehat{\mathbb{E}}[(|X| - c)^+] \leq C_V((|X| - c)^+) = \int_c^{\infty} \mathbb{V}(|X| \geq x) dx \rightarrow 0 \text{ as } c \rightarrow \infty.$$

And so

$$0 \leq \widehat{\mathbb{E}}[|X|] - \widehat{\mathbb{E}}[|X| \wedge c] \leq \widehat{\mathbb{E}}[(|X| - c)^+] \rightarrow 0 \text{ as } c \rightarrow \infty.$$

(3.12) is proved. \square

Proof of Theorems 3.3 of 3.5. We first prove (a) of Theorem 3.5. (a) of Theorem 3.3 follows from (a) of Theorem 3.5 because $\mathbb{V}^* = \mathbb{V}$ when \mathbb{V} is countably sub-additive.

Without loss of generality, we assume $\widehat{\mathbb{E}}[X_1] = 0$. Define

$$f_c(x) = (-c) \vee (x \wedge c), \quad \widehat{f}_c(x) = x - f_c(x) \quad (3.13)$$

and

$$\overline{X}_j = f_j(X_j) - \widehat{\mathbb{E}}[f_j(X_j)], \quad \overline{S}_j = \sum_{i=1}^j \overline{X}_i, \quad j = 1, 2, \dots$$

Then $f_c(\cdot), \widehat{f}_c(\cdot) \in C_{l,Lip}(\mathbb{R})$, and $\overline{X}_j, j = 1, 2, \dots$ are negatively dependent. Let $\theta > 1, n_k = [\theta^k]$. For $n_k < n \leq n_{k+1}$, we have

$$\begin{aligned} \frac{S_n}{n} &= \frac{1}{n} \left\{ \overline{S}_{n_{k+1}} + \sum_{j=1}^{n_{k+1}} \widehat{\mathbb{E}}[f_j(X_j)] + \sum_{j=1}^n \widehat{f}_j(X_j) - \sum_{j=n+1}^{n_{k+1}} f_j(X_j) \right\} \\ &\leq \frac{\overline{S}_{n_{k+1}}^+}{n_k} + \frac{\sum_{j=1}^{n_{k+1}} |\widehat{\mathbb{E}}[f_j(X_1)]|}{n_k} + \frac{\sum_{j=1}^{n_{k+1}} |\widehat{f}_j(X_j)|}{n_k} \\ &\quad + \frac{\sum_{j=n_{k+1}+1}^{n_{k+1}} \{f_j^+(X_j) - \widehat{\mathbb{E}}[f_j^+(X_j)]\}}{n_k} + \frac{\sum_{j=n_{k+1}+1}^{n_{k+1}} \{f_j^-(X_j) - \widehat{\mathbb{E}}[f_j^-(X_j)]\}}{n_k} \\ &\quad + \frac{(n_{k+1} - n_k) \widehat{\mathbb{E}}[X_1]}{n_k} \\ &=: (I)_k + (II)_k + (III)_k + (IV)_k + (V)_k + (VI)_k. \end{aligned}$$

It is obvious that

$$\lim_{k \rightarrow \infty} (VI)_k = (\theta - 1) \widehat{\mathbb{E}}[|X_1|] \leq (\theta - 1) C_V(|X_1|)$$

by Lemma 3.9 (b).

For $(I)_k$, applying (2.7) yields

$$\begin{aligned} \mathbb{V}(\overline{S}_{n_{k+1}} \geq \epsilon n_k) &\leq \frac{\sum_{j=1}^{n_{k+1}} \widehat{\mathbb{E}}[\overline{X}_j^2]}{\epsilon^2 n_k^2} \leq \frac{4 \sum_{j=1}^{n_{k+1}} \widehat{\mathbb{E}}[f_j^2(X_1)]}{\epsilon^2 n_k^2} \\ &\leq \frac{4n_{k+1}}{\epsilon^2 n_k^2} + \frac{4 \sum_{j=1}^{n_{k+1}} \widehat{\mathbb{E}}[(|X_1| \wedge j)^2]}{\epsilon^2 n_k^2}. \end{aligned}$$

It is obvious that $\sum_k \frac{n_{k+1}}{n_k^2} < \infty$. Also,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\sum_{j=1}^{n_{k+1}} \widehat{\mathbb{E}}[(|X_1| \wedge j)^2]}{n_k^2} &\leq \sum_{j=1}^{\infty} \widehat{\mathbb{E}}[(|X_1| \wedge j)^2] \sum_{k: n_{k+1} \geq j} \frac{1}{n_k^2} \\ &\leq C \sum_{j=1}^{\infty} \widehat{\mathbb{E}}[(|X_1| \wedge j)^2] \frac{1}{j^2} < \infty \end{aligned}$$

by Lemma 3.9 (a). Hence

$$\sum_{k=1}^{\infty} \mathbb{V}^*((I)_k \geq \epsilon) \leq \sum_{k=1}^{\infty} \mathbb{V}((I)_k \geq \epsilon) < \infty.$$

By the Borel-Cantelli lemma and the countable sub-additivity of \mathbb{V}^* , it follows that

$$\mathbb{V}^*\left(\limsup_{k \rightarrow \infty} (I)_k > \epsilon\right) = 0, \quad \forall \epsilon > 0$$

Similarly,

$$\mathbb{V}^*\left(\limsup_{k \rightarrow \infty} (IV)_k > \epsilon\right) = 0, \quad \mathbb{V}^*\left(\limsup_{k \rightarrow \infty} (V)_k > \epsilon\right) = 0, \quad \forall \epsilon > 0.$$

For $(II)_k$, note that by the (finite) sub-additivity,

$$|\widehat{\mathbb{E}}[f_j(X_1)]| = |\widehat{\mathbb{E}}[f_j(X_1)] - \widehat{\mathbb{E}}X_1| \leq \widehat{\mathbb{E}}[|f_j(X_1)|] = \widehat{\mathbb{E}}[(|X_1| - j)^+] \rightarrow 0.$$

It follows that

$$(II)_k = \frac{n_{k+1}}{n_k} \frac{\sum_{j=1}^{n_{k+1}} |\widehat{\mathbb{E}}[f_j(X_1)]|}{n_{k+1}} \rightarrow 0.$$

At last, we consider $(III)_k$. By the Borel-Cantelli Lemma, we will have

$$\mathbb{V}^*\left(\limsup_{k \rightarrow \infty} (III)_k > 0\right) \leq \mathbb{V}^*(\{|X_j| > j\} \text{ i.o.}) = 0$$

if we have shown that

$$\sum_{j=1}^{\infty} \mathbb{V}^*(|X_j| > j) \leq \sum_{j=1}^{\infty} \mathbb{V}(|X_j| > j) < \infty. \quad (3.14)$$

Let g_ϵ be a function satisfying that its derivatives of each order are bounded, $g_\epsilon(x) = 1$ if $x \geq 1$, $g_\epsilon(x) = 0$ if $x \leq 1 - \epsilon$, and $0 \leq g_\epsilon(x) \leq 1$ for all x , where $0 < \epsilon < 1$. Then

$$g_\epsilon(\cdot) \in C_{l,Lip}(\mathbb{R}) \quad \text{and} \quad I\{x \geq 1\} \leq g_\epsilon(x) \leq I\{x > 1 - \epsilon\}.$$

Hence, by (3.1),

$$\begin{aligned} \sum_{j=1}^{\infty} \mathbb{V}(|X_j| > j) &\leq \sum_{j=1}^{\infty} \widehat{\mathbb{E}}[g_{1/2}(|X_j|/j)] = \sum_{j=1}^{\infty} \widehat{\mathbb{E}}[g_{1/2}(|X_1|/j)] \quad (\text{since } X_j \stackrel{d}{=} X_1) \\ &\leq \sum_{j=1}^{\infty} \mathbb{V}(|X_1| > j/2) \leq 1 + C_{\mathbb{V}}(2|X_1|) < \infty. \end{aligned}$$

(3.14) is proved. So, we conclude that

$$\mathbb{V}^*\left(\limsup_{n \rightarrow \infty} \frac{S_n}{n} > \epsilon\right) = 0 \quad \forall \epsilon > 0,$$

by the arbitrariness of $\theta > 1$. Hence

$$\mathbb{V}^*\left(\limsup_{n \rightarrow \infty} \frac{S_n}{n} > 0\right) = \mathbb{V}^*\left(\bigcup_{k=1}^{\infty} \left\{\limsup_{n \rightarrow \infty} \frac{S_n}{n} > \frac{1}{k}\right\}\right) \leq \sum_{k=1}^{\infty} \mathbb{V}^*\left(\limsup_{n \rightarrow \infty} \frac{S_n}{n} > \frac{1}{k}\right) = 0.$$

Finally,

$$\mathbb{V}^* \left(\liminf_{n \rightarrow \infty} \frac{S_n}{n} < \widehat{\mathcal{E}}[X_1] \right) = \mathbb{V}^* \left(\limsup_{n \rightarrow \infty} \frac{\sum_{k=1}^n (-X_k - \widehat{\mathbb{E}}[-X_k])}{n} > 0 \right) = 0.$$

The proof of (3.2) is now completed.

For (b) of Theorems 3.3 and 3.5, suppose $C_{\mathbb{V}}(|X_1|) = \infty$. Then, by (3.1),

$$\begin{aligned} \sum_{j=1}^{\infty} \widehat{\mathbb{E}} \left[g_{1/2} \left(\frac{|X_j|}{Mj} \right) \right] &= \sum_{j=1}^{\infty} \widehat{\mathbb{E}} \left[g_{1/2} \left(\frac{|X_1|}{Mj} \right) \right] \quad (\text{since } X_j \stackrel{d}{=} X_1) \\ &\geq \sum_{j=1}^{\infty} \mathbb{V}(|X_1| > Mj) = \infty, \quad \forall M > 0. \end{aligned} \quad (3.15)$$

For any $l \geq 1$,

$$\begin{aligned} \nu \left(\sum_{j=1}^n g_{1/2} \left(\frac{|X_j|}{Mj} \right) < l \right) &= \nu \left(\exp \left\{ -\frac{1}{2} \sum_{j=1}^n g_{1/2} \left(\frac{|X_j|}{Mj} \right) \right\} > e^{-l/2} \right) \\ &\leq e^{l/2} \widehat{\mathcal{E}} \left[\exp \left\{ -\sum_{j=1}^n g_{1/2} \left(\frac{|X_j|}{Mj} \right) \right\} \right] = e^{l/2} \prod_{j=1}^n \widehat{\mathcal{E}} \left[\exp \left\{ -\frac{1}{2} g_{1/2} \left(\frac{|X_j|}{Mj} \right) \right\} \right] \end{aligned}$$

by (3.1) again and the independence because $0 \leq \exp \left\{ -\frac{1}{2} g_{1/2} \left(\frac{|x_j|}{Mj} \right) \right\} \in C_{l, Lip}(\mathbb{R})$. Applying the elementary inequality

$$e^{-x} \leq 1 - \frac{1}{2}x \leq e^{-x/2}, \quad \forall 0 \leq x \leq 1/2$$

yields

$$\widehat{\mathcal{E}} \left[\exp \left\{ -\frac{1}{2} g_{1/2} \left(\frac{|X_j|}{Mj} \right) \right\} \right] \leq 1 - \frac{1}{4} \widehat{\mathbb{E}} \left[g_{1/2} \left(\frac{|X_j|}{Mj} \right) \right] \leq \exp \left\{ -\frac{1}{4} \widehat{\mathbb{E}} \left[g_{1/2} \left(\frac{|X_j|}{Mj} \right) \right] \right\}.$$

It follows that

$$\nu \left(\sum_{j=1}^n g_{1/2} \left(\frac{|X_j|}{Mj} \right) < l \right) \leq \exp \left\{ -\frac{1}{4} \sum_{j=1}^n \widehat{\mathbb{E}} \left[g_{1/2} \left(\frac{|X_j|}{Mj} \right) \right] \right\} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

by (3.15). So

$$\mathbb{V} \left(\sum_{j=1}^n g_{1/2} \left(\frac{|X_j|}{Mj} \right) > l \right) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

If \mathbb{V} is continuous as assumed in Theorem 3.3, then $\mathbb{V} \equiv \mathbb{V}^*$. If $\widehat{\mathbb{E}}$ is countably sub-additive as assumed in Theorem 3.5, then

$$\mathbb{V}^*(|X| \geq c) \leq \mathbb{V}(|X| \geq c) \leq \widehat{\mathbb{E}}[g_{\epsilon}(|X|/c)] \leq \mathbb{V}^*(|X| \geq c(1-\epsilon)),$$

by (3.1) and (3.6). In either case, we have

$$\mathbb{V}^* \left(\sum_{j=1}^n g_{1/2} \left(\frac{|X_j|}{Mj} \right) > l/2 \right) \geq \mathbb{V} \left(\sum_{j=1}^n g_{1/2} \left(\frac{|X_j|}{Mj} \right) > l \right) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Now, by the continuity of \mathbb{V}^* ,

$$\mathbb{V}^* \left(\limsup_{n \rightarrow \infty} \frac{|X_n|}{n} > \frac{M}{2} \right) = \mathbb{V}^* \left(\left\{ \frac{|X_j|}{Mj} > \frac{1}{2} \right\} \text{ i.o.} \right) \geq \mathbb{V}^* \left(\sum_{j=1}^{\infty} g_{1/2} \left(\frac{|X_j|}{Mj} \right) = \infty \right)$$

$$= \lim_{l \rightarrow \infty} \mathbb{V}^* \left(\sum_{j=1}^{\infty} g_{1/2} \left(\frac{|X_j|}{M_j} \right) > l/2 \right) = \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{V}^* \left(\sum_{j=1}^n g_{1/2} \left(\frac{|X_j|}{M_j} \right) > l/2 \right) = 1.$$

On the other hand,

$$\limsup_{n \rightarrow \infty} \frac{|X_n|}{n} \leq \limsup_{n \rightarrow \infty} \left(\frac{|S_n|}{n} + \frac{|S_{n-1}|}{n} \right) \leq 2 \limsup_{n \rightarrow \infty} \frac{|S_n|}{n}.$$

It follows that

$$\mathbb{V}^* \left(\limsup_{n \rightarrow \infty} \frac{|S_n|}{n} > m \right) = 1, \quad \forall m > 0.$$

Hence

$$\mathbb{V}^* \left(\limsup_{n \rightarrow \infty} \frac{|S_n|}{n} = +\infty \right) = \lim_{m \rightarrow \infty} \mathbb{V}^* \left(\limsup_{n \rightarrow \infty} \frac{|S_n|}{n} > m \right) = 1,$$

which contradict (3.3) and (3.8). So, $C_V(|X_1|) < \infty$.

Finally, we consider (c) of Theorem 3.3. For (3.4), we first show that

$$\mathbb{V} \left(\frac{S_n}{n} > \widehat{\mathbb{E}}[X_1] - \epsilon \right) \rightarrow 1, \quad \forall \epsilon > 0. \quad (3.16)$$

Let $f_c(x)$ and $\widehat{f}_c(x)$ be defined as in (3.13). Then

$$\mathbb{V} \left(\frac{|\sum_{k=1}^n \widehat{f}_c(X_k)|}{n} > \epsilon \right) \leq \frac{\sum_{k=1}^n \widehat{\mathbb{E}}[|\widehat{f}_c(X_k)|]}{\epsilon n} \leq \frac{\widehat{\mathbb{E}}[(|X_1| - c)^+]}{\epsilon} \rightarrow 0 \quad \text{as } c \rightarrow \infty,$$

$\widehat{\mathbb{E}}[X_1] - \widehat{\mathbb{E}}[f_c(X_1)] \rightarrow 0$ as $c \rightarrow \infty$, and by Theorem 2.3,

$$\begin{aligned} \mathcal{V} \left(\frac{\sum_{k=1}^n f_c(X_k)}{n} \leq \widehat{\mathbb{E}}[f_c(X_1)] - \epsilon \right) &= \mathcal{V} \left(\sum_{k=1}^n (-f_c(X_k) - \widehat{\mathcal{E}}[-f_c(X_k)]) \geq n\epsilon \right) \\ &\leq \frac{\widehat{\mathcal{E}} \left[\left| \left(\sum_{k=1}^n (-f_c(X_k) - \widehat{\mathcal{E}}[-f_c(X_k)]) \right)^+ \right|^2 \right]}{n^2 \epsilon^2} \leq 2 \frac{\widehat{\mathbb{E}} \left[(-f_c(X_1) - \widehat{\mathcal{E}}[-f_c(X_1)])^2 \right]}{n \epsilon^2} \\ &\leq \frac{2(2c)^2}{n \epsilon^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Then (3.16) is proved. By considering $\{-X_n; n \geq 1\}$ instead, from (3.16) we have

$$\mathbb{V} \left(\frac{S_n}{n} \leq \widehat{\mathcal{E}}[X_1] + \epsilon \right) \rightarrow 1, \quad \forall \epsilon > 0. \quad (3.17)$$

Note the independence. We conclude that

$$\begin{aligned} &\mathbb{V} \left(\frac{S_n}{n} < \widehat{\mathcal{E}}[X_1] + \epsilon \quad \text{and} \quad \frac{S_{n^2} - S_n}{n^2 - n} > \widehat{\mathbb{E}}[X_1] - \epsilon \right) \\ &\geq \widehat{\mathbb{E}} \left[\phi \left(\frac{S_n}{n} - \widehat{\mathcal{E}}[X_1] \right) \phi \left(\widehat{\mathbb{E}}[X_1] - \frac{S_{n^2} - S_n}{n^2 - n} \right) \right] \\ &\geq \widehat{\mathbb{E}} \left[\phi \left(\frac{S_n}{n} - \widehat{\mathcal{E}}[X_1] \right) \right] \cdot \widehat{\mathbb{E}} \left[\phi \left(\widehat{\mathbb{E}}[X_1] - \frac{S_{n^2} - S_n}{n^2 - n} \right) \right] \\ &\geq \mathbb{V} \left(\frac{S_n}{n} < \widehat{\mathcal{E}}[X_1] + \frac{\epsilon}{2} \right) \cdot \mathbb{V} \left(\frac{S_{n^2} - S_n}{n^2 - n} > \widehat{\mathbb{E}}[X_1] - \frac{\epsilon}{2} \right) \rightarrow 1, \quad \forall \epsilon > 0, \end{aligned}$$

where $\phi(x) \in C_{l,Lip}(\mathbb{R})$ is a function such that $I\{x \leq \epsilon\} \geq \phi(x) \geq I\{x \leq \epsilon/2\}$. Now, by (3.2) and the continuity of \mathbb{V} ,

$$\mathbb{V} \left(\liminf_{n \rightarrow \infty} \frac{S_n}{n} \leq \widehat{\mathcal{E}}[X_1] + \epsilon \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{S_n}{n} \geq \widehat{\mathbb{E}}[X_1] - \epsilon \right)$$

$$\begin{aligned}
&\geq \mathbb{V} \left(\liminf_{n \rightarrow \infty} \frac{S_n}{n} \leq \widehat{\mathcal{E}}[X_1] + \epsilon \text{ and } \limsup_{n \rightarrow \infty} \frac{S_{n^2} - S_n}{n^2 - n} \geq \widehat{\mathbb{E}}[X_1] - \epsilon \right) \\
&\geq \mathbb{V} \left(\frac{S_n}{n} < \widehat{\mathcal{E}}[X_1] + \epsilon \text{ and } \frac{S_{n^2} - S_n}{n^2 - n} > \widehat{\mathbb{E}}[X_1] - \epsilon \text{ i.o.} \right) \\
&\geq \limsup_{n \rightarrow \infty} \mathbb{V} \left(\frac{S_n}{n} < \widehat{\mathcal{E}}[X_1] + \epsilon \text{ and } \frac{S_{n^2} - S_n}{n^2 - n} > \widehat{\mathbb{E}}[X_1] - \epsilon \right) = 1, \quad \forall \epsilon > 0.
\end{aligned}$$

By the continuity of \mathbb{V} again,

$$\mathbb{V} \left(\liminf_{n \rightarrow \infty} \frac{S_n}{n} \leq \widehat{\mathcal{E}}[X_1] \text{ and } \limsup_{n \rightarrow \infty} \frac{S_n}{n} \geq \widehat{\mathbb{E}}[X_1] \right) = 1,$$

which, together with (3.2) implies (3.4).

Finally, note

$$\frac{S_n}{n} - \frac{S_{n-1}}{n-1} = \frac{X_n}{n} - \frac{S_{n-1}}{n-1} \frac{1}{n} \rightarrow 0 \text{ a.s. } \mathbb{V}.$$

It can be verified that (3.4) implies (3.5). \square

Proof of Corollary 3.4. It is sufficient to note the facts that $\mathbb{V}(A) = \widehat{\mathbb{E}}[I_A]$ is continuous in $\mathcal{H} = \{A, I_A \in \mathcal{H}\}$ and all events we consider are in \mathcal{H} because \mathcal{H} is monotone and $I\{x \geq 1\} = \lim_{\epsilon \rightarrow 0} g_\epsilon(x)$. \square

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